Math 246B Lecture 21 Notes

Daniel Raban

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1 Bloch's Theorem and Range of Meromorphic Functions

1.1 Bloch's theorem

We want to prove the following theorem.

Theorem 1.1 (Picard's little theorem)**.** *Let $f \in \mathrm{Hol}(\mathbb{C})$ be entire and nonconstant. Then the range $f(\mathbb{C})$ omits at most 1 point of $\mathbb{C}$.*

Remark 1.1. It is possible for the range to omit one point. $e^z \neq 0$ for all $z \in \mathbb{C}$.

Remark 1.2. There exists a topological proof of this fact, but it requires the machinery of covering spaces, so we will not visit it at this time.

Proposition 1.1. *Let $f \in \mathrm{Hol}(|z| < 1)$ be such that $f(0) = 0$, $f'(0) = 1$. If, furthermore, $|f| \leq M$, then $f(\{|z| < 1\}) \supseteq D(0, 1/(4M))$.*

We will write $D := \{|z| < 1\}$.

Remark 1.3. If $M = 1$, then $f(D) = D$ by Schwarz's lemma.

Proof. Let $w \in \mathbb{C} \setminus f(D)$. Then $w \neq 0$, the function $1 - f/w \neq 0$ in D, and $1 = f/w = 1$ at $z = 0$. Then there exists $g \in \mathrm{Hol}(|z| < 1)$ such that $g^2 = 1 - f/w$ and $g(0) = 1$. Differentiate and let $z = 0$ to get $2g(0)g'(0) = -1/w$. So $g'(0) = -1/(2w)$, which gives the Taylor expansion

$$g(z) = 1 - \frac{z}{2w} + \cdots .$$

Now given $h \in \mathrm{Hol}(|z| < 1)$, we have $h = \sum_{n=0}^{\infty} a_n z^n$ and

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\varphi})|^2 \, d\varphi = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

for $r < 1$. In particular, apply this property to g. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\varphi})|^2 \, d\varphi \leq \|g\|_{\infty}^2 \leq 1 + \frac{M}{|w|}$$

and

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \geq 1 + \frac{r^2}{4|w|^2}.$$

Sending $r \to 1$, we get $1/(4|w|^2) \leq M/|w|$. That is, $|w| \geq 1/(4M)$. $\square$

Theorem 1.2 (A. Bloch)**.** *There exists an absolute constant $\ell > 0$ such that if $f \in$ $\mathrm{Hol}(|z| < 1)$ and $f'(0) = 1$, then the range of $f(D)$ contains an open disc of radius ℓ.*

Proof. Assume first that f is holomorphic near $|z| \leq 1$. Let $\mathrm{Aut}(D)$ be the set of holomorphic bijections $\varphi : D \to D$; this is the set of automorphisms of D:

$$\varphi \in \mathrm{Aut}(D) \iff \varphi(z) = \lambda \frac{z - \alpha}{1 - \overline{\alpha} z},$$

where $|\lambda| = 1$, and $\alpha \in D$. We have

$$(1 - |z|^2)|\varphi'(z)| = 1 - |\varphi(z)|^2$$

for all $\varphi \in \mathrm{Aut}(D)$. Define $B(f, z) = (1 - |z|^2)|f'(z)|$ when $z \in D$. For any $\varphi \in \mathrm{Aut}(D)$,

$$B(f \circ \varphi, z) = (1 - |z|^2)|f'(\varphi(z))||\varphi'(z)| = (1 - |\varphi(z)|)^2|f'(\varphi(z))| = B(f, \varphi(z)).$$

The function $B(f, \cdot)$is continuous in D, nonnegative, and equal to 0 on ∂D. Let $a \in D$ be such that B achieves its maximum at a.

Assume first that $a = 0$. Then $|f'(z)| \leq 1/(1 - |z|^2)$ for $|z| < 1$ for $|z| < 1$. We get

$$|f(z) - f(0)| = \left| \int_0^1 \frac{d}{dt} f(tz)\, dt \right| \leq \frac{|z|}{1 - |z|^2}, \qquad |z| < 1.$$

If $|z| \leq R < 1$, we get

$$|f(z) - f(0)| \leq \frac{R}{1 - R^2} = M.$$

Apply the previous proposition to $(f(Rz) - f(0))/R$, which is a holomorphic function bounded by M/R. Then $f(D)$ contains an open disc of radius $R\frac{1}{4(M/R)} = R^2/(4M) = R(1 - R)^2/4$. This is true for any $0 < R < 1$, so the optimal choice of R is $\sqrt{3}/3$. The corresponding radius is $\sqrt{3}/18$.

In general, we may have $a \neq 0$. Let $\psi \in \mathrm{Aut}(D)$ be such that $\psi(0) = a$. Consider $g = f \circ \psi$. Then

$$B(g, z) = B(f, \psi(z)) \leq B(f, a) = B(g, 0),$$

by pulling back using ψ and the conformal invariance of B. Note that the right hand side equals $|g'(0)|$, so $|g'(0)| \geq 1$. The previous discussion can be applied to the function $(g(Rz) - g(0))/(Rg'(0))$. So the $g(D)$ contains an open disc of radius $\sqrt{3}18|g'(0)| \geq \sqrt{3}18$. Since $g(D) = f(D)$. we get the result, if f is holomorphic near $|z| \leq 1$.

In general, let $f_\rho(z) = (1/\rho)f(\rho z)$, where $0 < \rho < 1$. Then $f_\rho(D)$ contains a fixed disc. Then $f(D) \supseteq \rho f_\rho(D)$, which contains a disc of radius $\rho\sqrt{3}/18$. Pick any such ρ to get the theorem. $\square$

1.2 Range of meromorphic functions

We will use Bloch's theorem to prove Picard's little theorem next time. Here is a corollary of Picard's theorem.

Corollary 1.1. *Let f be meromorphic in $\mathbb{C}$ and nonconstant. Then f assumes all values in $\mathbb{C}$ with at most 2 exceptions.*

Proof. Assume f does not take on the distinct values $a, b, c \in \mathbb{C}$. Let $g(z) = 1/(f(z) - c)$. This is holomorphic away form the poles of f. The singularities at the poles of f are removable for g, so g can be extended to an entire holomorphic function. Its range omits 2 values: $1/(a - c)$ and $1/(b - c)$. So g is constant by Picard's little theorem. $\square$

Example 1.1. Let

$$f(z) = \frac{1}{e^z + 1}.$$

This function omits the values $0, 1$.

Example 1.2. Suppose we try to solve $f^n + g^n = 1$ with $n \geq 3$. This equation has no nonconstant solution by this corollary to Picard's little theorem.